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The Classification of Displacements and Rotations in a Cellular Space—Time

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Abstract

The implications of the lack of continuity in space-time brought about by every observation process are discussed in connection with the need for finding new observables based on realisable measuring processes. Displacements in a cellular space are defined and their rules of combination are formulated. Rotations in a cellular space are defined in terms of a set of radius displacements and in terms of a set of instructions which tell how these radii should be used.

1. *Introduction*

In a previous paper (Cole, 1971a) it was pointed out that whenever we use a space-time description in the observation and tabulation of physical processes, the space-time always has a cellular structure brought about by the imperfect resolving power of the measuring apparatus we use. It was also suggested that it takes an infinite amount of energy (or cost) in order to be able to refine the apparatus sufficiently in order to be able to obtain a description of events within a continuous space-time structure. Thus the space-time which we use is always cellular, but the exact cellular structure is observer-dependent. Consistent with the amount of energy an observer has for refining his apparatus he is able to use any cellular structure he pleases, not necessarily a simple uniform cubic lattice structure, although in practice this is one of the easiest to use. Many attempts (Das, 1960; Bopp, 1967; Cole, 1970, 1971b) at forming theories based on a cellular space-time structure then involve the introduction of a set of fundamental parameters into existing continuous equations, but in the case of a simple rectangular lattice these parameters are not to be regarded as the lengths of the cell edges because the usual idea of length is not defined in a cellular space. These parameters must be regarded as observer-dependent quantities which enable us formally to make the equations discrete. In the same way if the

existing equations contain concepts which are not defined in a cellular space-time scheme, for example momentum and angular momentum, then it would be inconsistent to make these equations discrete while at the same time retaining these concepts. As Bohm *et al.* (1970) have pointed out, it is not sufficient to merely make our equations discrete, we must also develop new observable quantities to replace those which were developed specifically for the continuum.

Since every method of observation imposes its own cellular structure on space-time then there can be no answer to the question of whether or not there is *really* continuity beneath the cellular structure because no observation process will enable us to see or demonstrate this continuity. Of course in our theories we are at liberty to use a set of four continuous coordinates $(x_1,\ldots,x_4)$ to represent space-time, but it must then be realised that these quantities are not observable, and are merely devices which may make our theories easier to use. However, the calculations involved in many of our theories, whether they are eigenvalue calculations (Bassett, 1968) or the gravitational n-body problem (Miller & Prendergast, 1968) now require the use of a computer, and immediately this happens we lose any benefits which the use of continuity might have given. It is then very wasteful of time and effort to retain the device of continuity in our equations, only to have to make those equations discrete in order to be able to solve them.

Thus we require new equations for the description and prediction of observable quantities, but first we must decide what these quantities really are. We are no longer allowed to talk of the observables as things in themselves—concepts which exist whether or not we make measurements, but they must be defined by the way in which they are measured and are therefore related directly to each individual measuring process. In this way, no meaning can be attached to the statement that there are many different ways of measuring the *same* observable, but rather each of these processes defines its own observable. An attempt to show how the idea of distance can be modified and how eigenvalues can be interpreted in the light of this idea has been given by Cole (1971a).

At this stage it is convenient to make a few basic definitions. Space-time will consist of a basic cell structure, the cells of which we will assume do not overlap. There is nothing to stop the observer using a set of overlapping cells so that in describing the position of an event he may use one of several cells, but to make his description less complicated he will not do this. The cells may be put into one–one correspondence with the integers, and the integers m, n, p, q, etc. will be used to label the cells of the basic structure. The set of cells in this basic structure will be denoted by S. The subsets of S, denoted by M, N, P, Q, etc., will define regions denoted by $\bar{M}$, $\bar{N}$, $\bar{P}$ and $\bar{Q}$, etc., which are themselves cells in some coarser cellular structure. So in particular for a given cell n, $\overline{\{n\}} \equiv n$. The set of all possible regions $\bar{M}$ formed from the basic structure will be denoted by $\mathscr{S}$, so that $S \subset \mathscr{S}$. Furthermore, since actual experiments always take place in finite regions (Antoine & Gleit, 1971), both of these sets will contain a finite number of elements.

Because of the imperfect resolving power of the apparatus used in making observations, an event can be located by specifying in which cell it occurs, but it is impossible to locate an event more precisely within the cell. The event may have such an extension that it takes more than one cell to describe its location and in that case it will be said to be located in some element of $\mathscr{S}$. In this way regions of $\mathscr{S}$ are treated in exactly the same way as the basic cells of S from which they are composed, and there is no fundamental distinction between cell and region, as there is between point and plane in a continuous geometry.

As pointed out earlier the idea of distance which was developed in a continuous space is not defined in a cellular space. A modification of this concept based on a real measuring process has been suggested by Cole (1971a). These measurements are made using a measuring rod r which is laid to coincide with the regions $\bar{M}$ and $\bar{N}$ of $\mathscr{S}$ between which the 'distance' is to be measured. The rod r will itself have a cellular structure dictated by the amount of energy the observer has available for refining it, and the observer then tabulates the pairs of cells of the rod which coincide with the regions $\bar{M}$ and $\bar{N}$. From each pair of these cells a single value is then extracted and the set of these single values for the given regions $\bar{M}$ and $\bar{N}$, denoted by $d_r(\bar{M}, \bar{N})$ and called the distance set, is then taken to represent the 'distance' between the regions. The values of the elements for a given pair $\bar{M}$ and $\bar{N}$ will then depend on the cellular structure of r, that is, on the observer, and again these sets will have a finite number of elements because of the finite extension of the rod. For example, in the special case in which the rod has a uniform cubic structure specified by the parameter ξ the distance set could be taken to have the form

$$d_r(\bar{M}, \bar{N}) = \{K\xi : K \text{ integer}, K_1(\bar{M}, \bar{N}) \leqslant K \leqslant K_2(\bar{M}, \bar{N})\}$$

As the cellular structure is refined further to the ideal continuous limit we should expect the elements of $d_r(\bar{M}, \bar{N})$ to all converge to one value which will be the usual distance between the two points which are the limits of the subdivisions of the regions $\bar{M}$ and $\bar{N}$.

In general we will consider only those d-set constructions which give the following properties to the d-sets. For all $\bar{L}$, $\bar{M}$ and $\bar{N}$ in $\mathscr{S}$,

(a) all elements of $d_r(\bar{L}, \bar{M})$ are real, non-negative and distinct;

(b) $d_r(\bar{L}, \bar{M}) = d_r(\bar{M}, \bar{L})$;

(c) $0 \in d_r(\bar{L}, \bar{L})$;

(d) $\min d_r(\bar{L}, \bar{N}) \leqslant \max d_r(\bar{L}, \bar{M}) + \max d_r(\bar{M}, \bar{N})$;

(e) $d_r(\bar{L}, \bar{M}) = \bigcup_{l \in L} \bigcup_{m \in M} d_r(l, m)$.

The problem of transformations between the cellular space-times of two different observers was also tackled by Cole (1971a). It was found that the transformation is not as exact as in the continuous case, with at best a cell in one space being associated with a set of cells in the other. These sets

depend on the information passed between the observers concerning their placements of common events within their frameworks. The transformation becomes more exact, in the sense that these sets become smaller, as more information of this type is exchanged.

A similar problem arises when one comes to consider the idea of rotation in a cellular space, either about a region or about an axis. Of course, one first has to define an axis in a cellular space. To crystallise the problem consider the following situation. A rod is thrown α times into a cellular space and the positions of the regions of the rod which suitably define its ends are noted for each throw. It is found that the ends fall into the pairs of regions $(\bar{M}_1, \bar{N}_1)$, $(\bar{M}_2, \bar{N}_2)$, . . ., $(\bar{M}_\alpha, \bar{N}_\alpha)$. If the rod is now placed with one end in a region $\bar{M}$, what are the possible regions into which the other end falls? The answer is that these regions $\bar{N}$ are such that

$$d_r(\bar{M}, \bar{N}) \supset \bigcap_{i=1}^{\alpha} d_r(\bar{M}_i, \bar{N}_i)$$

for all measuring rods r. The set of all possible regions $\bar{N}$ for this to be so then defines a shell centred on the region $\bar{M}$. As detailed in Section 3 this example points to the possibility of a general classification of different types of rotation in a cellular space.

The remainder of the paper is concerned with classification rather than prediction because before we can write down equations of motion we must derive physically realisable concepts for the equations to handle. The next section deals with displacements in a cellular space and Section 3 deals with the classification of the possible rotations allowable in such a space.

2. *Displacements in a Cellular Space*

For any two regions $\bar{M}$ and $\bar{N}$ in $\mathscr{S}$ we can define a quantity called a *cellor* and denoted by $\bar{M}\bar{N}$ which can be thought of as representing a displacement from $\bar{M}$ into $\bar{N}$, analogous to the notion of a vector in continuous theory. The magnitude of $\bar{M}\bar{N}$, denoted by $|\bar{M}\bar{N}|_r$, is defined as the observer-dependent distance set between $\bar{M}$ and $\bar{N}$:

$$|\bar{M}\bar{N}|_r \equiv d_r(\bar{M}, \bar{N}) \tag{2.1}$$

Thus the magnitude of a cellor depends on the measuring rod used in the measurement. In what follows we will consider only one measuring rod unless otherwise stated so that the suffix r will be removed.

The conditions (b), (c) and (e) of Section 1 then give

$$|\bar{M}\bar{N}| = |\bar{N}\bar{M}| \tag{2.2}$$

$$0 \in |\bar{L}\bar{L}| \text{ for all } \bar{L} \text{ in } \mathscr{S} \tag{2.3}$$

$$|\bar{M}\bar{N}| = \bigcup_{m \in M} \bigcup_{n \in N} |mn| \tag{2.4}$$

Certain cellors can be combined to form a third cellor, and addition of cellors is defined in the sense that for all $\bar{L}$, $\bar{M}$ and $\bar{N}$ in $\mathscr{S}$,

$$\bar{L}\bar{N} = \bar{L}\bar{M} + \bar{M}\bar{N}$$

and $\bar{L}\bar{M} + \bar{P}\bar{Q}$ is defined if and only if $M \equiv P$. Clearly addition is defined only between certain pairs of cellors and when it is defined the operation is not always commutative. But it is associative, for

$$\bar{L}\bar{M} + (\bar{M}\bar{N} + \bar{N}\bar{P}) = \bar{L}\bar{P} = (\bar{L}\bar{M} + \bar{M}\bar{N}) + \bar{N}\bar{P}$$

Condition (d) of Section 1 then gives

$$\min |\bar{L}\bar{M} + \bar{M}\bar{N}| \leqslant \max |\bar{L}\bar{M}| + \max |\bar{M}\bar{N}| \tag{2.5}$$

In particular, for all $\bar{L}$ and $\bar{M}$ in $\mathscr{S}$,

$$\begin{aligned} \bar{L}\bar{L} + \bar{L}\bar{M} &= \bar{L}\bar{M} \\ \bar{L}\bar{M} + \bar{M}\bar{M} &= \bar{L}\bar{M} \end{aligned} \tag{2.6}$$

so that $\bar{L}\bar{L}$ and $\bar{M}\bar{M}$ respectively can be called the left and right zero cellors of $\bar{L}\bar{M}$. Also,

$$\bar{L}\bar{L} + \bar{L}\bar{L} = \bar{L}\bar{L}$$

and (2.3) shows that the zero element is contained in the magnitude of every zero cellor. However, if the magnitude of a cellor contains zero, it is not necessarily a zero cellor.

In fact, let $\mathscr{C} \equiv \{\bar{M}\bar{N} : \bar{M} \text{ and } \bar{N} \in \mathscr{S}\}$. Then it can be easily checked that $\mathscr{C}$ is a Brandt groupoid. [For the properties of a Brandt groupoid see Jacobson (1943.)]

As a further possible classification of these displacements we can say that $\bar{M}\bar{N}$ is an *expansion cellor* if $M \subset N$, and $\bar{P}\bar{Q}$ is a *contraction cellor* if $P \supset Q$. These constructions could possibly be of use when describing the expansions and contractions of gases.

Clearly in a general cellular structure the concept of parallel displacement does not exist. The idea exists only when the structure possesses some degree of regularity, and only then can we talk of two non-coincident cellors being 'equal'. In this paper we will deal only with the properties of a general cellular space so that all cellors will be considered as localised, and the idea of regularity in the structure will not be pursued.

3. *Rotations*

In order to discuss rotations in a cellular space it is necessary to look at the nature of the steps involved when considering rotations in a continuous space, and then to generalise each of these steps. In continuous space we start with two points O and P and then when we talk of a rotation of the radius OP about O we mean that the line OP is moved into OP' where $|OP'| = |OP|$. Thus in talking of a rotation of OP about O we first fix the

radius $|OP|$ and then use the instruction 'rotate to OP' where OP' and OP have the same length'. The set of all possible points P' such that this is so then forms a shell about O of radius $|OP|$.

The situation in a cellular space is clearly more complicated. If we are to talk about the rotation of a cellor $\bar{M}\bar{N}$ about $\bar{M}$, ending up with a shell of regions, then because $|\bar{M}\bar{N}|$ contains a number of values rather than one single value there are many arbitrary instructions which we may impose on the rotation process. For example, we may demand that $\bar{M}\bar{N}$ is rotated into $\bar{M}\bar{N}'$ where $|\bar{M}\bar{N}'| = |\bar{M}\bar{N}|$ with the corresponding shell composed of all possible regions $\bar{N}'$ such that this is so, but clearly in a general cellular space this shell may be empty. Again we may demand that $\bar{M}\bar{N}$ is rotated into $\bar{M}\bar{N}'$ where $|\bar{M}\bar{N}| \cap |\bar{M}\bar{N}'| \neq \varnothing$, and this instruction then defines another shell. As an added complication the shell may eventually depend on a set of radii as is illustrated by the example of the rod in Section 1. Thus the shells of the theory will depend firstly on a set of radii and secondly on how these radii are to be used. In order that this idea of rotation corresponds to that which is already used in continuous theory, we will require that all shells defined using these separate instructions tend to the shell we get in a continuous theory when the appropriate limit is taken. This is achieved by imposing certain restrictions on the instructions used in forming the shells.

A. The radius proposition. The basic instruction will be introduced through the *radius proposition*, denoted by $F_\alpha(x;a_1,\ldots,a_\alpha)$, which is a proposition linking any set of cellors $x, a_1, \ldots, a_\alpha$ in $\mathscr{C}$ $(\alpha = 1,2,\ldots)$ such that

RPI. $F_\alpha(x;x,\ldots,x)$ is true for all x in $\mathscr{C}$, for all $\alpha \geqslant 1$;

RPII. $F_\alpha(x;a_1,\ldots,a_\alpha) \to |x|_r \cap |a_i|_r \neq \varnothing \quad (i = 1,\ldots,\alpha)$ for all measuring rods r.

Let $\Phi_\alpha(x;a_1,\ldots,a_\alpha)$ be the proposition that $|x| \cap |a_i| \neq \varnothing$ $(i = 1,\ldots,\alpha)$ for all r. Then Φ_α is itself a radius proposition and for all radius propositions F_α, $F_\alpha \to \Phi_\alpha$.

Special cases can be outlined as follows. We say that F_α is *scalar* if $F_\alpha(x;a_1,\ldots,a_\alpha) \equiv F_\alpha(|x|;|a_1|,\ldots,|a_\alpha|)$, that is, the proposition F_α links only the distance sets of the cellors involved. We say that F_α is *factorisable* if $F_\alpha(x;a_1,\ldots,a_\alpha) \leftrightarrow F_1(x;a_1)$ and $F_1(x;a_2)$ and $\ldots$ $F_1(x;a_\alpha)$.

Now for each $\bar{M} \in \mathscr{S}$ define

$$R_{\bar{M}}(F_\alpha;a_1,\ldots,a_\alpha) \equiv \{\bar{N}: F_\alpha(\bar{M}\bar{N};a_1,\ldots,a_\alpha) \text{ is true}\} \tag{3.1}$$

called the $(F_\alpha;a_1,\ldots,a_\alpha)$-shell centred on $\bar{M}$. In this way the shells depend not only on the set of radii but also on the instruction contained in the statement of F_α. The following results can then be easily proved:

(a) If F_α is factorisable then

$$R_{\bar{M}}(F_\alpha;a_1,\ldots,a_\alpha) = \bigcap_{i=1}^{\alpha} R_{\bar{M}}(F_1;a_i) \tag{3.2}$$

and so

$$R_{\bar{M}}(F_\alpha; a_1, \ldots, a_\alpha) \supset R_{\bar{M}}(F_\beta; a_1, \ldots, a_\beta) \qquad \text{for } \beta \geqslant \alpha$$

(b) If F_α is scalar and $\bar{N} \in R_{\bar{M}}(F_\alpha; a_1, \ldots, a_\alpha)$ then $\bar{M} \in R_{\bar{N}}(F_\alpha; a_1, \ldots, a_\alpha)$. This shows that if $\bar{N}$ lies on the scalar shell about $\bar{M}$ then $\bar{M}$ lies on the corresponding shell about $\bar{N}$.

(c) If F_α and G_α are radius propositions such that $G_\alpha \to F_\alpha$ then for all $\bar{M} \in \mathscr{S}$,

$$R_{\bar{M}}(G_\alpha; a_1, \ldots, a_\alpha) \subset R_{\bar{M}}(F_\alpha; a_1, \ldots, a_\alpha)$$

In particular for all radius propositions G_α,

$$R_{\bar{M}}(G_\alpha; a_1, \ldots, a_\alpha) \subset R_{\bar{M}}(\Phi_\alpha; a_1, \ldots, a_\alpha)$$

(d) For all $\bar{M}$ and $\bar{N}$ in $\mathscr{S}$ and all radius propositions F_α,

$$\bar{N} \in R_{\bar{M}}(F_\alpha; \bar{M}\bar{N}, \bar{M}\bar{N}, \ldots, \bar{M}\bar{N}) \tag{3.3}$$

We can now illustrate this with some examples of different forms for $F_\alpha(x; a_1, \ldots, a_\alpha)$. Each example will be scalar and specified by a superscript on F_α.

1. $F_\alpha{}^1 \equiv \Phi_\alpha : |x| \cap |a_i| \neq \varnothing \ (i = 1, \ldots, \alpha)$ for all measuring rods. This is factorisable.

2. $F_\alpha{}^2 : |x| \supset \bigcap_{i=1}^{\alpha} |a_i|$ for all measuring rods. The answer to the question posed in Section 1 concerning the end of the rod is that the end could be in any element of $R_{\bar{M}}(F_\alpha{}^2; \bar{M}_1\bar{N}_1, \ldots, \bar{M}_\alpha\bar{N}_\alpha)$. In fact the $\bar{M}_i\bar{N}_i$ must be such that

$$\bar{N}_i \in R_{\bar{M}_i}(F^2_{\alpha-1}; \bar{M}_1\bar{N}_1, \ldots, \bar{M}_{i-1}\bar{N}_{i-1}, \bar{M}_{i+1}\bar{N}_{i+1}, \ldots, \bar{M}_\alpha\bar{N}_\alpha), \qquad (i = 1, \ldots, \alpha)$$

The result

$$R_{\bar{M}}(F_\beta{}^2; \bar{M}_1\bar{N}_1, \ldots, \bar{M}_\beta\bar{N}_\beta) \supset R_{\bar{M}}(F_\alpha{}^2; \bar{M}_1\bar{N}_1, \ldots, \bar{M}_\alpha\bar{N}_\alpha) \qquad \text{for } \beta \geqslant \alpha$$

means that as more results $\bar{M}_{\alpha+1}\bar{N}_{\alpha+1}, \ldots, \bar{M}_\beta\bar{N}_\beta$ of initial throws of the rod are tabulated then more information is available for predicting the possible regions into which the end of the rod falls. This extra information enables us to predict smaller regions for the end of the rod and these smaller regions are then included in the $R_{\bar{M}}$ set to make it larger.

3. $F_\alpha{}^3 : |x| \subset \bigcap_{i=1}^{\alpha} |a_i|$ for all measuring rods. This is factorisable.

4. $F_\alpha{}^4 : |x| = \bigcap_{i=1}^{\alpha} |a_i|$ for all measuring rods.

5. $F_\alpha{}^5 : |x| \supset |a_i| \ (i = 1, \ldots, \alpha)$ for all measuring rods. This is factorisable.

6. $F_\alpha{}^6 : |x| = |a_i| \ (i = 1, \ldots, \alpha)$ for all measuring rods. This is factorisable.

Note that from these examples the sequences $F_\alpha^{\,6} \to F_\alpha^{\,5} \to F_\alpha^{\,2} \to F_\alpha^{\,1}$ and $F_\alpha^{\,6} \to F_\alpha^{\,4} \to F_\alpha^{\,3} \to F_\alpha^{\,1}$ emerge. This suggests the possibility of defining general sequences of radius propositions with Φ_α as limit. This idea will not be taken further in this paper.

B. Analysis of F_1. For all a and b in $\mathscr{C}$, define the proposition $F_1^{\,x}$ by

$$F_1^{\,x}(b;a) \equiv F_1(a;b) \tag{3.4}$$

Then special cases may be classified in the following way: We say that F_1 is *symmetric* if for all a and b in $\mathscr{C}$

$$F_1^{\,x}(a;b) = F_1(a;b) \tag{3.5}$$

we say that F_1 is *transitive* if for all a, b and c in $\mathscr{C}$

$$F_1(a;b) \text{ and } F_1(b;c) \to F_1(a;c) \tag{3.6}$$

It can then be shown that (i) if F_1 is a radius proposition then so is $F_1^{\,x}$, (ii) if F_1 is a transitive radius proposition then so is $F_1^{\,x}$, and (iii) $(F_1^{\,x})^x = F_1$.

For example, Φ_1 is symmetric, $F_1^{\,2} \equiv F_1^{\,5}$ is transitive, $F_1^{\,3}$ is transitive, $F_1^{\,4} \equiv F_1^{\,6}$ is symmetric and transitive, and $(F_1^{\,2})^x \equiv F_1^{\,3}$.

The following general results can then be easily proved. For all radius propositions F_1 and all $\bar{M}$ and $\bar{N}$ in $\mathscr{S}$,

(a) $\bar{Q} \in R_{\bar{M}}(F_1; \bar{M}\bar{N})$ if and only if $\bar{N} \in R_{\bar{M}}(F_1^{\,x}; \bar{M}\bar{Q})$.

(b) If F_1 is transitive and $\bar{Q} \in R_{\bar{M}}(F_1; \bar{M}\bar{N})$, then

$$R_{\bar{M}}(F_1; \bar{M}\bar{Q}) \subset R_{\bar{M}}(F_1; \bar{M}\bar{N})$$
$$R_{\bar{M}}(F_1^{\,x}; \bar{M}\bar{Q}) \supset R_{\bar{M}}(F_1^{\,x}; \bar{M}\bar{N})$$

(c) In particular if F_1 is transitive and symmetric and $\bar{Q} \in R_{\bar{M}}(F_1; \bar{M}\bar{N})$, then

$$R_{\bar{M}}(F_1; \bar{M}\bar{Q}) = R_{\bar{M}}(F_1; \bar{M}\bar{N})$$

Clearly when considering rotations within a continuous space we deal with only one radius proposition equivalent to $F_1^{\,6}$ which is both symmetric and transitive, and so if a point P_1 lies on the shell through a point P_2 about some point O then the two shells through P_1 and P_2 about O will coincide. This is result (c), but this result does not hold for every rotation in a cellular space.

C. Rotations about axes. Perpendicularity. Any cellor $\bar{M}_1\bar{M}_2$ in $\mathscr{C}$ can be taken to represent an axis in a cellular space, so that an axis is specified by naming in order its two end regions. A *circle* about this axis can be defined as the intersection of two shells which are centred on $\bar{M}_1$ and $\bar{M}_2$. We say that

$$C_{\bar{M}_1\bar{M}_2}(F_\alpha; a_1, \ldots, a_\alpha | G_\beta; b_1, \ldots, b_\beta) \equiv R_{\bar{M}_1}(F_\alpha; a_1, \ldots, a_\alpha) \cap R_{\bar{M}_2}(G_\beta; b_1, \ldots, b_\beta) \tag{3.7}$$

defines an $(F_\alpha; a_1, \ldots, a_\alpha | G_\beta; b_1, \ldots, b_\beta)$-circle about $\bar{M}_1\bar{M}_2$.

We then have the following results which follow directly from (3.7), (3.2) and (3.3). For all $\bar{M}_1$ and $\bar{M}_2$ in $\mathscr{S}$ and all $a_1, \ldots, b_\beta$ in $\mathscr{C}$,

(a) $C_{\bar{M}_1\bar{M}_2}(F_\alpha; a_1, \ldots, a_\alpha | G_\beta; b_1, \ldots, b_\beta) = C_{\bar{M}_2\bar{M}_1}(G_\beta; b_1, \ldots, b_\beta | F_\alpha; a_1, \ldots, a_\alpha)$.

(b) If F_α is factorisable, then

$$C_{\bar{M}_1\bar{M}_2}(F_\alpha; a_1, \ldots, a_\alpha | G_\beta; b_1, \ldots, b_\beta) = \bigcap_{i=1}^{\alpha} C_{\bar{M}_1\bar{M}_2}(F_1; a_i | G_\beta; b_1, \ldots, b_\beta)$$

and

$$C_{\bar{M}_1\bar{M}_2}(F_\alpha; a_1, \ldots, a_\alpha | G_\beta; b_1, \ldots, b_\beta) \supset C_{\bar{M}_1\bar{M}_2}(F_\gamma; a_1, \ldots, a_\gamma | G_\beta; b_1, \ldots, b_\beta) \qquad \text{for all } \gamma \geqslant \alpha$$

(c) For each $\bar{N}$ in $\mathscr{S}$,

$$\bar{N} \in C_{\bar{M}_1\bar{M}_2}(F_\alpha; \bar{M}_1\bar{N}, \ldots, \bar{M}_1\bar{N} | G_\beta; \bar{M}_2\bar{N}, \ldots, \bar{M}_2\bar{N})$$

this ensures that $\bar{N}$ is itself contained in the circle about $\bar{M}_1\bar{M}_2$ which is the intersection of the shells with radii $\bar{M}_1\bar{N}$ and $\bar{M}_2\bar{N}$.

It is now possible to discuss the idea of perpendicularity in a cellular space, in as much as such a relation exists between a cellor $\bar{M}_1\bar{M}_2$ and any cellor $\bar{N}_1\bar{N}_2$ such that $\bar{N}_1$ and $\bar{N}_2$ both lie in the same circle about $\bar{M}_1\bar{M}_2$. Thus the idea of perpendicularity must be qualified by specifying the nature of the circle in terms of radii and radius propositions.

We say that for any $\bar{N}_1$ and $\bar{N}_2$ in $C_{\bar{M}_1\bar{M}_2}(F_\alpha; a_1, \ldots, a_\alpha | G_\beta; b_1, \ldots, b_\beta)$, $\bar{N}_1\bar{N}_2$ is $(F_\alpha; a_1, \ldots, a_\alpha | G_\beta; b_1, \ldots, b_\beta)$—perpendicular to $\bar{M}_1\bar{M}_2$, denoted by

$$\bar{N}_1\bar{N}_2 \perp (F_\alpha; a_1, \ldots, a_\alpha | G_\beta; b_1, \ldots, b_\beta)\, \bar{M}_1\bar{M}_2$$

Then denoting the brackets above by $(A|B)$, we have $\bar{N}_2\bar{N}_1 \perp (A|B)\,\bar{M}_1\bar{M}_2$ if and only if $\bar{N}_1\bar{N}_2 \perp (A|B)\,\bar{M}_1\bar{M}_2$ if and only if $\bar{N}_1\bar{N}_2 \perp (B|A)\,\bar{M}_2\bar{M}_1$.

The relation $\perp (A|B)$ is left-transitive in the sense that

$$\bar{N}_1\bar{N}_2 \perp (A|B)\,\bar{M}_1\bar{M}_2 \quad \text{and} \quad \bar{N}_2\bar{N}_3 \perp (A|B)\,\bar{M}_1\bar{M}_2 \rightarrow \bar{N}_1\bar{N}_3 \perp (A|B)\,\bar{M}_1\bar{M}_2$$

and is right-transitive in the sense that

$$\bar{N}_1\bar{N}_2 \perp (A|B)\,\bar{M}_1\bar{M}_2 \quad \text{and} \quad \bar{N}_1\bar{N}_2 \perp (B|C)\,\bar{M}_2\bar{M}_3 \rightarrow \bar{N}_1\bar{N}_2 \perp (A|C)\,\bar{M}_1\bar{M}_3$$

Conversely if we are given the three regions $\bar{M}_1$, $\bar{M}_2$ and $\bar{N}$ and we wish to construct a cellor at $\bar{N}$ which is perpendicular to $\bar{M}_1\bar{M}_2$, we first look for all those brackets $(A|B)$ for which $\bar{N} \in C_{\bar{M}_1\bar{M}_2}(A|B)$. If $(A_1|B_1)$ is such a bracket then for any $\bar{Q} \in C_{\bar{M}_1\bar{M}_2}(A_1|B_1)$, $\bar{N}\bar{Q}$ is a cellor perpendicular to $\bar{M}_1\bar{M}_2$ such that $\bar{N}\bar{Q} \perp (A_1|B_1)\,\bar{M}_1\bar{M}_2$.

Note that symmetry could be introduced into the relation $\perp (A|B)$ [that is, for each a and b in $\mathscr{C}$ and each $(A|B)$ there exists $(A'|B')$ such that $a \perp (A|B)b \rightarrow b \perp (A'|B')a$] only if the idea of parallel displacement is introduced when there is some degree of regularity in the structure.

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